Quiz 9 - In Class Right Now - Due 7:40 AM Fri 11/10/23

Lesson 29 - Extrema of functions of 2 variables

I. Second Partial Derivative Test

II. Examples

I. Second Partial Derivative Test

Rcl: 2nd Deriv. Test (for fctns of 1 var.)

If $f'(c) = 0$ =0 and $f''(x)$ is cont$\bar{s}$ on an interval containing c. Then

(1) $f''(c) > 0 \Rightarrow f$ has min @ $x = c$

(2) $f''(c) < 0 \Rightarrow f$ has max @ $x = c$

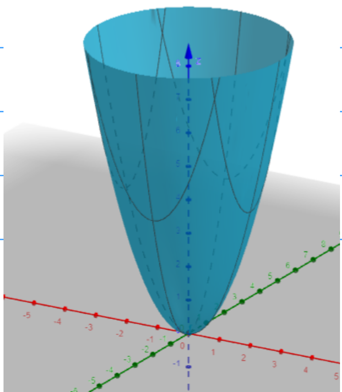
(3) $f''(c) = 0$ means test is inconclusive (i.e. - USELESS INFO.)

There is an analogue for multivariable fctns

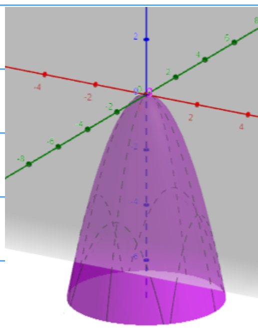
1st: 2 definitions and some pics

(D) (c,d) is a critical point of $f(x,y)$ if $f_x(c,d) = 0$ AND $f_y(c,d) = 0$

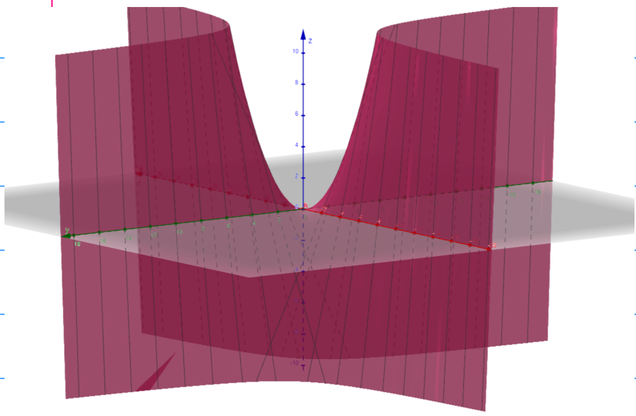
Geogebra pictures



local min @ (0,0,0)



local max @ (0,0,0)



$(0,0,0)$ is a
saddle point

CU on some cross sections
CD on other cross sections

(D) The discriminant of $f(x,y)$ is

$$D(x,y) \overset{\text{def'n}}{=\!=} f_{xx}(x,y)\, f_{yy}(x,y) - (f_{xy}(x,y))^2$$

2nd Partial Derivatives Test

Let (a,b) be a critical point for $f(x,y)$. Then

(1) $D(a,b) > 0$ and $f_{xx}(a,b) > 0 \Rightarrow$ f has relative min @ (a,b)

(2) $D(a,b) > 0$ and $f_{xx}(a,b) < 0 \Rightarrow$ f has relative max @ (a,b)

(3) $D(a,b) < 0 \Rightarrow f$ has a saddle point @ (a,b)

(4) $D(a,b) = 0$ Then test is inconclusive (i.e. Useless Info.)

II. Examples

$\boxed{\text{Ex}}$ Find and classify the critical pts if possible.

(OpenStax)

a) $g(x,y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

$g_x(x,y) = 2x + 2y + 4$
$g_y(x,y) = 2x - 8y - 6$

Crit Pts: $\begin{cases} 2x + 2y + 4 = 0 \\ 2x - 8y - 6 = 0 \end{cases}$ $\begin{cases} 2x + 2y = -4 & ① \\ 2x - 8y = 6 & ② \end{cases}$

① − ②

$$\begin{aligned} 2x + 2y &= -4 \\ -2x + 8y &= -6 \\ \hline 10y &= -10 \\ \boxed{y = -1} \end{aligned}$$

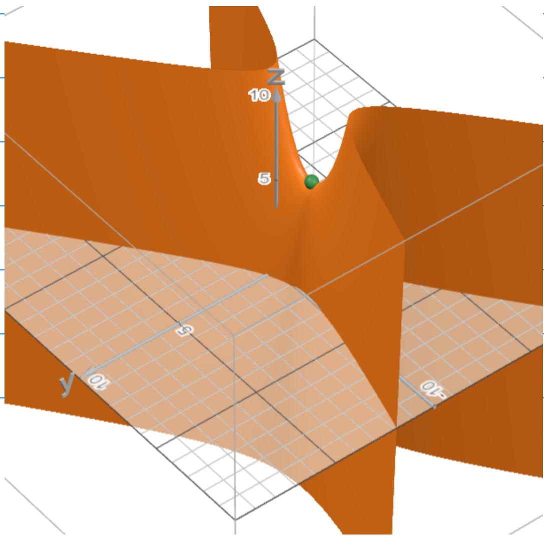
$$\begin{aligned} 2x + 2(-1) &= -4 \\ 2x - 2 &= -4 \\ 2x &= -2 \\ \boxed{x = -1} \end{aligned}$$

Crit pt: $(-1, -1)$

$g_{xx}(x,y) = 2$ $\quad g_{yy}(x,y) = -8$ $\quad g_{xy}(x,y) = 2$

$$\begin{aligned} D(-1,-1) &= g_{xx}(-1,-1)\, g_{yy}(-1,-1) - \left(g_{xy}(-1,-1)\right)^2 \\ &= 2(-8) - 2^2 \\ &= -16 - 4 \\ &= -20 < 0 \end{aligned}$$

$\xrightarrow{\text{2nd D. test}}$ f has a saddle point @ $(-1,-1)$



From Desmos 3D

b) (Openstax)

$f(x,y) = \frac{1}{3}x^3 + y^2 + 2xy - 6x - 3y + 4$

$f_x(x,y) = x^2 + 2y - 6$
$f_y(x,y) = 2y + 2x - 3$

Crit Pts:

$\begin{cases} x^2 + 2y - 6 = 0 \\ 2y + 2x - 3 = 0 \end{cases}$ $\begin{cases} x^2 + 2y = 6 & (1) \\ 2x + 2y = 3 & (2) \end{cases}$

(2) $2y = 3 - 2x \to (1)$ $x^2 + (3 - 2x) = 6$

$x^2 - 2x + 3 - 6 = 0$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3 \quad x = -1$

Back to $2y = 3 - 2x$

$x = 3$: $2y = 3 - 2(3)$
$2y = -3$
$y = -3/2$

$x = -1$: $2y = 3 - 2(-1)$
$2y = 5$
$y = 5/2$

Crit pts: $(3, -3/2)$ $(-1, 5/2)$

z vals: $\frac{-29}{4}$ $\frac{41}{12}$

$f_x(x,y) = x^2 + 2y - 6$
$f_y(x,y) = 2y + 2x - 3$

$f_{xx}(x,y) = 2x$
$f_{yy}(x,y) = 2$
$f_{xy}(x,y) = 2$

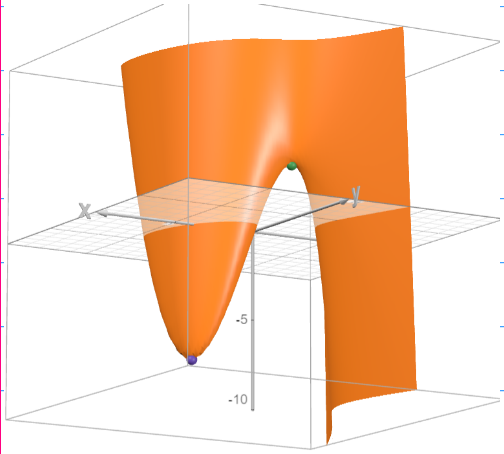
$D(x,y) = (2x)(2) - 2^2 = 4x - 4$

$D(3, -3/2) = 4(3) - 4 = 8 > 0$ $f_{xx}(3, -3/2) = 2(3) = 6 > 0$

local min @ $(3, -3/2)$

$D(-1, \tfrac{5}{2}) = 4(-1) - 4 = -8 < 0$

Saddle pt @ $(-1, \tfrac{5}{2})$



Desmos 3D graph